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INTEGRAL EQUATIONS FOR A THIN INCLUSION IN A HOMOGENEOUS ELASTIC MEDIUM†

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A study is made of equilibrium in a homogeneous elastic medium containing a thin inclusion whose elastic moduli differ substantially from those of the medium. The solution depends on two non-dimensional parameters: the ratio δ_1 of the characteristic linear dimensions of the inclusion and the ratio δ_2 of the elastic moduli of the inclusion and the medium. While δ_1 is always small, δ_2 may be either small or large. The problem of constructing the principal asymptotic terms of the elastic fields in the neighbourhood of a thin inhomogeneity based on these parameters has been reduced [1] to the solution of integral (pseudodifferential) equations on the middle surface of the inclusion. Similar equations are obtained with two-dimensional models of thin inclusions [2–5]. Some properties of the solutions of these equations will be discussed below. A method is proposed for the numerical solution of the equations, based on introducing a special class of approximating functions, thanks to which the problem can be reduced to a system of linear algebraic equations whose matrix can be calculated by analytical means. The idea of the method is due to V. G. Maz'ya.

1. INTEGRAL EQUATIONS FOR THIN DEFORMABLE AND RIGID INCLUSIONS

A HOMOGENEOUS elastic medium with tensor of moduli C_0 contains an inclusion that occupies a bounded region V, one of whose characteristic dimensions h is small compared with the other two (of order l), so that $\delta_1 = h/l$ is a small parameter. The inclusion is ideally connected to the medium

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along its boundary; its elastic properties are defined by a tensor of moduli C. We wish to solve the problem of elastostatics for the medium with the inclusion under a given external load.

For applied problems of the mechanics of composites we are particularly concerned with thin inclusions whose elastic moduli differ substantially from those of the medium. In such situations the parameter δ_2, defined as the ratio between the characteristic moduli of the inclusion and the medium $[\delta_2 = O(CC_0^{-1})]$ is either small (compliant inclusions) or large (rigid or stiff inclusions).

We might remark that the most valuable information concerning the elastic fields in the neighbourhood of thin inclusions is provided by the principal terms of the asymptotic expansions of the fields in terms of δ_1 and δ_2. To construct these terms one has to find a solution of the problem in the limit as $\delta_1 \to 0$, $\delta_2 \to 0$ (or $\delta_2 \to \infty$) with the quotient δ_1/δ_2 remaining fixed, equal to its value for the inclusion. This asymptotic solution describes the elastic fields at distances from the inclusion surface that exceed its characteristic transverse dimension h; it is of particular interest when one is formulating criteria for the brittle fracture of bodies with inclusions [2], dealing with the average properties of media with several thin inclusions, etc.

We will begin with thin compliant inclusions, for which δ_2 is small. We shall assume that the middle surface Ω of the inclusion is a Lyapunov surface, bounded by a closed contour Γ, $n(x)$ is the continuous vector field of the normal, defined on Ω and $x = (x_1, x_2, x_3)$ is a typical point of the medium. It has been shown [1] that the principal terms of the asymptotic expansions in terms of δ_1 and δ_2 of the strain field $\varepsilon(x)$ and the stress field $\sigma(x)$ in a medium with a thin deformable inclusion are as follows:

$$\varepsilon_{\alpha\beta}(x) = \varepsilon_{0\alpha\beta}(x) + \int_\Omega K_{\alpha\beta\lambda\mu}(x-x')\, C_0^{\lambda\mu\nu\rho} n_\nu(x')\, b_\rho(x')\, d\Omega'$$

$$\sigma^{\alpha\beta}(x) = \sigma_0^{\alpha\beta}(x) + \int_\Omega S^{\alpha\beta\lambda\mu}(x-x')\, n_\lambda(x')\, b_\mu(x')\, d\Omega'$$

$$K_{\alpha\beta\lambda\mu}(x) = -[\nabla_\alpha \nabla_\lambda G_{\beta\mu}(x)]_{(\alpha\beta)(\lambda\mu)} \tag{1.1}$$

$$S^{\alpha\beta\lambda\mu}(x) = C_0^{\alpha\beta\nu\rho} K_{\nu\rho\tau\delta}(x)\, C_0^{\tau\delta\lambda\mu} - C_0^{\alpha\beta\lambda\mu}\delta(x)$$

where $\varepsilon_0(x)$ and $\sigma_0(x)$ are the external fields of strains and stresses which would exist in the medium were it not for the inhomogeneity and the external loads, $G(x)$ is Green's function for a homogeneous medium with moduli C_0 and $\delta(x)$ is a three-dimensional delta-function. We know [6] that $G(x)$ is an even homogeneous function of degree -1 whose Fourier transform is

$$G^*(k) = L^{-1}(k), \quad L^{\alpha\beta}(k) = k_\lambda C_0^{\lambda\alpha\beta\mu} k_\mu$$

The vector field $b(x)$—the density of the potentials in (1.1)—satisfies the following equation on Ω [1]:

$$\lambda^{\alpha\beta}(x)\, b_\beta(x) + \int_\Omega T^{\alpha\beta}(x, x')\, b_\beta(x')\, d\Omega' = n_\beta(x)\, \sigma_0^{\alpha\beta}(x), \quad x \in \Omega$$

$$\lambda^{\alpha\beta}(x) = h^{-1}(x)\, n_\lambda(x)\, C^{\lambda\alpha\beta\mu} n_\mu(x), \quad T^{\alpha\beta}(x, x') = -n_\lambda(x)\, S^{\lambda\alpha\beta\mu}(x-x')\, n_\mu(x') \tag{1.2}$$

where $h(x)$ is the transverse dimension of the inclusion along the normal $n(x)$ to Ω at $x \in \Omega$.

The reader should note that the operator T may be written only formally as an integral operator with kernel $T(x, x')$, because the integral in question diverges for $x \in \Omega$ for arbitrarily smooth functions $b(x)$ ($T(x, x') \sim |x - x'|^{-3}$ as $x' \to x$). It can be shown [7] that T is a pseudodifferential operator with a smooth homogeneous symbol—a homogeneous function of degree 1. It has also been shown [7, 8] that T admits of a regular representation for functions $b(x)$ that have a continuous derivative along Ω. It follows from the general theory of equations of type (1.2) [9] that the additional condition $b(x) = 0$ for $x \in \Gamma$ will ensure uniqueness of the solution.

It can be shown [1] that if $b(x)$ satisfies condition (1.2), the strain and stress fields (1.1) correspond to a solution of the following boundary-value problem of elasticity theory: determine the vector field of displacements $u(x)$ by solving the Lamé equations for a homogeneous medium with moduli C_0 and prescribed external load, on the assumption that the following conditions hold on Ω:

$$[u_\alpha(x)] = b_\alpha(x), \quad [n_\beta(x)\sigma^{\alpha\beta}(x)] = 0, \quad n_\beta(x)\sigma^{\alpha\beta}(x) = \lambda^{\alpha\beta}(x)\, b_\beta(x) \tag{1.3}$$

where $[f(x)]$ denotes the jump in $f(x)$ across Ω at a point x in the direction of the normal, and $\lambda(x)$ has the form of (1.2). A heuristic approach proposed in [2, 3] to the solution of the thin inclusion problem involves replacing the inclusion with its middle surface on the assumption that boundary conditions of type (1.3) are satisfied. If $\lambda(x)=0$, conditions (1.3), and therefore also the representations (1.1) and Eq. (1.2), describe a crack in a homogeneous elastic medium.

We will now consider the case of stiff inclusions—large δ_2. The principal terms of the asymptotic expansion of the strain and stress fields (the limits of $\varepsilon(x)$ and $\sigma(x)$ as $\delta_1 \to 0$, $\delta_2 \to \infty$, $\delta_1\delta_2 = O(1)$) will then be

$$\varepsilon_{\alpha\beta}(x) = \varepsilon_{0\alpha\beta}(x) + \int_\Omega K_{\alpha\beta\lambda\mu}(x-x')\, q^{\lambda\mu}(x')\, d\Omega'$$
$$\sigma^{\alpha\beta}(x) = \sigma_0^{\alpha\beta}(x) + \int_\Omega S^{\alpha\beta\lambda\mu}(x-x')\, C^{-1}_{0\lambda\mu\nu\rho} q^{\nu\rho}(x')\, d\Omega' \tag{1.4}$$

where $q(x)$ is the tensor of the surface Ω:

$$n_\alpha(x)\, q^{\alpha\beta}(x) = 0, \quad \Theta_{\lambda\mu}{}^{\alpha\beta}(x)\, q^{\lambda\mu}(x) = q^{\alpha\beta}(x) \tag{1.5}$$

$\theta(x)$ is the orthogonal projection onto the tangent plane to Ω at x:

$$\Theta(x) = \Theta(n) = E_1 - 2E_5(n) + E_6(n), \quad n = n(x)$$
$$E_{1\alpha\beta\lambda\mu} = \delta_{\alpha(\lambda}\delta_{\beta)\mu}, \quad E_{5\alpha\beta\lambda\mu}(n) = n_{(\alpha}\delta_{\beta)(\lambda}n_{\mu)}, \quad E_{6\alpha\beta\lambda\mu}(n) = n_\alpha n_\beta n_\lambda n_\mu$$

and $\delta_{\alpha\beta}$ is the Kronecker delta.

The field $q(x)$ satisfies the following equation in Ω:

$$\mu_{\alpha\beta\lambda\mu}(x)\, q^{\lambda\mu}(x) + \int_\Omega U_{\alpha\beta\lambda\mu}(x, x')\, q^{\lambda\mu}(x')\, d\Omega' = \Theta_{\alpha\beta}^{\lambda\mu}(x)\, \varepsilon_{0\lambda\mu}(x) \tag{1.6}$$
$$\mu_{\alpha\beta\lambda\mu}(x) = h^{-1}(x)\, \Theta_{\alpha\beta}^{\nu\rho}(x)\, C^{-1}_{\nu\rho\tau\delta}\Theta^{\tau\delta}_{\lambda\mu}(x), \quad U_{\alpha\beta\lambda\mu} = \Theta_{\alpha\beta}^{\nu\rho}(x)\, K_{\nu\rho\tau\delta}(x-x')\, \Theta^{\tau\delta}_{\lambda\mu}(x')$$

The general operator U with kernel $U(x,x')$ is a pseudodifferential operator with principal homogeneous symbol—a homogeneous function of degree 1. A regular representation of this operator, over continuously differentiable functions $q(x)$ along Ω, has been established [1]. Equation (1.6) has a unique solution in the class of functions such that $e_\alpha(x)q^{\alpha\beta}(x) = 0$ on Γ, where $e(x)$ is the normal to Γ in the tangent plane to Ω at x.

Using the properties of the potentials on the right of Eqs (1.4), it can be shown [1] that, if the density $\mu(x)$ satisfies Eq. (1.6), then the fields $\varepsilon(x)$ and $\sigma(x)$ correspond to the following boundary-value problem of elasticity theory: solve the Lamé equations for a homogeneous medium with moduli C_0 under a given external load, on the assumption that the following boundary conditions are satisfied on the surface Ω (indices are omitted for simplicity):

$$[u(x)] = 0, \quad [\Theta(x)\varepsilon(x)] = 0, \quad \Theta(x)\varepsilon(x) = \mu(x)q(x) \tag{1.7}$$

where $\mu(x)$ is of the form (1.6) and $q(x)$ is the tensor of the surface Ω (1.5) and satisfies the following condition on Ω:

$$\partial_\alpha q^{\alpha\beta}(x) = -[n_\alpha(x)\sigma^{\alpha\beta}(x)], \quad \partial_\alpha = \nabla_\alpha - n_\alpha(x)n^\beta(x)\nabla_\beta \tag{1.8}$$

Here ∂_α is the gradient along Ω. The right-hand side of the first equality in (1.8) is the jump of the stress vector across Ω.

It has been shown [1] that the components of $q(x)$ have the meaning of integral stresses (forces) acting across sections of the thin inclusion, and then (1.8) is the equilibrium equation for these forces. A similar equation holds for the forces in a thin elastic shell in a torque-free stressed state [10]. Thus, conditions (1.7) and (1.8) describe a torque-free elastic shell (membrane) in contact with a homogeneous elastic medium. The first two equations of (1.7) should then be understood as compatibility conditions for the strains of the inclusion and the medium, while the last equation is Hooke's law for the inclusion. If $\mu = 0$, Eqs (1.7) and (1.8) are boundary conditions for an inextensible membrane sealed into a homogeneous elastic medium. In the two-dimensional problem conditions similar to (1.7) and (1.8) have been proposed to simulate thin rectilinear inclusions in an elastic plane [4].

2. ASYMPTOTIC BEHAVIOUR OF EQS (1.2) AND (1.6) AT THE EDGE OF THE INCLUSION

With regard to the numerical solution of Eqs (1.2) and (1.6) it is useful to have information about the structure of the solution near the boundary of the surface Ω. We will therefore consider the asymptotic behaviour of the solutions near a smooth part of the boundary contour Γ. We will rewrite the equations in symbolic form as

$$\lambda(x)b(x)+(Tb)(x)=n(x)\sigma_0(x),$$
$$\mu(x)q(x)+(Uq)(x)=\Theta(x)\varepsilon_0(x) \tag{2.1}$$

Let us assume that the transverse dimension $h(x)$ of the inclusion is a smooth bounded function which may be expressed in the neighbourhood of the boundary of the surface Ω in the form

$$h(x)=h_0(x_0)r^\gamma+O(r^{\gamma+1}),\quad \gamma>0 \tag{2.2}$$

where r is the distance from $x\in\Omega$ to $x_0\in\Omega$ along the normal to Γ and $h_0(x_0)$ is a smooth function on Γ. We will consider the asymptotic behaviour of continuous bounded solutions of Eqs (2.1) in the neighbourhood of a smooth part of Γ for such functions $h(x)$. It follows from the general theory of elliptic pseudodifferential equations [9] that the solutions of Eqs (2.1) near Γ are identical in the asymptotic limit with the solutions of the following model problems. Introduce a local Cartesian system of coordinates at $x_0\in\Gamma$, say y_1, y_2, y_3, with the y_2 axis directed along the tangent to Γ at x_0, the y_3 axis along the limiting normal to Ω at x_0; then the y_1 axis lies in the tangent plane to Ω at x_0. The model problems require the solution of Eqs (2.1) in the half-plane ($y_3=0$, $-\infty<y_2<\infty$, $y_1\geqslant 0$), with the right-hand sides dependent only on y_1 and the functions $\lambda(y)$ and $\mu(y)$ given by

$$\lambda(y)=\lambda_0 y_1^{-\gamma},\quad \lambda_0=h_0^{-1}(x_0)n_0Cn_0$$
$$\mu(y)=\mu_0 y_1^{-\gamma},\quad \mu_0=h_0^{-1}(x_0)\Theta(n_0)C^{-1}\Theta(n_0),\quad n_0=n(x_0) \tag{2.3}$$

The solutions of Eqs (2.1) will then also depend only on y_1. The asymptotic behaviour of these solutions as $y_1\to 0$ and of the solutions of Eqs (2.1) as $x\to x_0\in\Omega$ are the same [9].

Let us consider the model equation corresponding to a stiff inclusion (compliant inclusions were discussed in [8]). Substituting (2.3) into the second equation of (2.1), integrating with respect to y_2 and noting that $q(y)=q(y_1)$, we obtain the following equation ($y_1=t$):

$$t^{-\gamma}q(t)+U_0\int_0^\infty \frac{q(t')}{(t-t')^2}dt'=f(t),\quad t>0 \tag{2.4}$$

$$U_0=\mu_0^{-1}\int_{-\infty}^{\infty}\Theta(n_0)K(1,y_2,0)\Theta(n_0)dy_2,\quad f(t)=\mu_0^{-1}\Theta(n_0)\varepsilon_0$$

where $K(x_1,x_2,x_3)$ is the kernel of the integral operator in the first equation of (1.1).

The integral operator in (2.4) is defined for the continuously differentiable bounded function $q(t)$ by the formula [8]

$$\int_0^\infty \frac{q(t')}{(t-t')^2}dt'=\int_0^\infty \frac{q(t')-q(t)}{(t-t')^2}dt'-\frac{1}{t}q(t)$$

where the integral on the right should be understood in the sense of the Cauchy principal value.

To determine the asymptotic behaviour of the solution of Eq. (2.4) as $t\to 0$ we can use the results of [8], where a similar equation was considered. It turns out that the form of the asymptotic expansion depends on the form of the inclusion boundary, that is, on the exponent γ in formula

(2.2) for $h(x)$. If the boundary is "blunt" ($0 \leq \gamma < 1$), the behaviour of the solutions of Eq. (2.4) as $t \to 0$ is described by

$$q(t) = q_0 t^{1/2} + O(t^{1/2-\gamma})$$

If the boundary has a cusp ($\gamma > 1$), the asymptotic behaviour is described by the relation

$$q(t) = q_0 t^{\gamma} + O(t^{2\gamma-1})$$

For a cuspidal inclusion ($\gamma = 1$),

$$q(t) = q_1 t^{-s_1} + q_2 t^{-s_2} + O(t^{-s_3})$$

where s_1 and s_2 are the roots in the strip $-1 < \operatorname{Re} s < -1/2$ of the transcendental equation

$$\det[\Theta(n_0) + s\pi \operatorname{ctg}(s\pi) U_0] = 0 \tag{2.5}$$

The tensors $\theta(n_0)$ and U_0 are defined in (1.5) and (2.4) and s_3 is a root of the same equation in the strip $-2 < \operatorname{Re} s < -3/2$. For isotropic media and inclusions the exponents s_1, s_2 and s_3 are the roots of independent equations

$$\operatorname{tg} s\pi = -\frac{7(2-\varkappa_0)}{8\xi} s, \quad \operatorname{tg} s\pi = -\frac{s}{\xi}, \quad \xi = \frac{\mu_0}{n_0 \mu}, \quad \varkappa_0 = \frac{\lambda_0 + \mu_0}{\lambda_0 + 2\mu_0}$$

which follow from (2.5). Here λ_0 and μ_0 are the Lamé coefficients of the medium and μ is the shear modulus of the inclusion. Similar results have been obtained for compliant inclusions [2, 8].

3. NUMERICAL SOLUTION OF EQS (1.2) AND (1.6)

The class of inclusions for which Eqs (1.2) and (1.6) can be solved by analytical means is exhausted by thin ellipsoidal inclusions in a polynomial external field [8]. If the inclusion is not ellipsoidal, one must resort to numerical methods. In that case it is quite useful to treat the solution of Eqs (1.2) and (1.6) in a variational setting.

Let us consider T and U in Eqs (1.2) and (1.6) as operators in the Hilbert space $L_2(\Omega) = H(\Omega)$ [9]. We may assume that T and U are defined in a dense subspace of $L_2(\Omega)$—the space $C_0^\infty(\Omega)$ of infinitely differentiable compact-supported functions with support in the interior of Ω. For such functions the action of T and U is defined by the regularization formulas (2.6) and (2.14) of [1]. It can be shown that T and U are symmetric and positive definite, that is,

$$(Tb, b) \geq 0, \quad (Uq, q) \geq 0$$
$$\left((f, \varphi) = \int_\Omega f(x)\varphi(x)\, d\Omega, \ f, \varphi \in L_2(\Omega) \right)$$

where the equality will hold only if $b = 0$ and $q = 0$. That T has this property has been proved [7]; the proof for U is analogous. The operators $T_{(\lambda)}$ and $U_{(\mu)}$ in (1.2) and (1.6) ($T_{(\lambda)} = \lambda + T$, $U_{(\mu)} = \mu + U$) differ from T and Y by positive definite terms and are therefore also positive. Hence it follows [11] that the solutions of Eqs (1.2) and (1.6) minimize the functionals

$$F_{(\lambda)}(b) = \int_\Omega (T_{(\lambda)}^{\alpha\beta} b_\beta) b_\alpha\, d\Omega - 2 \int_\Omega n_\alpha \sigma_0^{\alpha\beta} b_\beta\, d\Omega$$

$$F_{(\mu)}(q) = \int_\Omega (U_{(\mu)\alpha\beta\lambda\mu} q^{\lambda\mu}) q^{\alpha\beta}\, d\Omega - 2 \int_\Omega \varkappa_{\alpha\beta}{}^{\lambda\mu} \varepsilon_{0\lambda\mu} q^{\alpha\beta}\, d\Omega$$

Consequently, $b(x)$ and $q(x)$ may be constructed by direct variational methods. The variational setting has been used [12] to solve a crack problem ($\lambda = 0$). The validity of an analogue of the finite element method, based on a variational approach, for crack problems has been established in [13, 14].

Another way to solve Eqs (1.2) and (1.6) employs a scheme usually applied to boundary integral equations in elasticity theory [15]. The surface Ω is divided into N disjoint domains Ω_1, so that $\Omega = \cup_i \Omega_i$. The solutions are approximated by linear combinations of standard functions with unknown coefficients within each Ω_i. Substituting the approximation into the initial equation and requiring the equation to hold true at a finite number of nodes, one obtains a system of linear algebraic equations for the coefficients. The problems involved in the implementation of this technique to solve crack problems in elastic media were discussed in [7, 16–18].

The main difficulties arise in computing the matrix coefficients of the above-mentioned system of linear algebraic equations, since the coefficients near the diagonal are integrals of rapidly varying functions over the domains Ω_i. In three dimensions, a sufficiently accurate determination of such integrals requires a large volume of computations, even when the approximating functions in Ω_i are as simple as possible (piecewise constant).

4. A SPECIAL CLASS OF APPROXIMATING FUNCTIONS

We will now introduce a class of approximating functions, by means of which the solution of Eqs (1.2) and (1.6) can be reduced to a system of linear algebraic equations with an analytic matrix.

Let Ω be a plane domain in $\mathbf{R}^3$ or a straight-line segment in $\mathbf{R}^2$. The kernels $T(x, x')$ and $U(x, x')$ of T and U will then depend solely on the difference between the arguments, so the operators themselves may be treated as convolution operators defining $b(x)$ and $q(x)$ as zero outside Ω. If b and q are of class $S(\mathbf{R}^n)$ ($n = 1, 2$) (infinitely differentiable functions that tend to zero as $|x|\to\infty$ faster than any power of $|x|^{-1}$), then T and U can be defined by the formulas

$$
\begin{aligned}
(Tb)(x) &= \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} T^*(k)\,b^*(k)\,e^{-ikx}\,dk \\
(Uq)(x) &= \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} U^*(k)\,q^*(k)\,e^{-ikx}\,dk
\end{aligned}
\tag{4.1}
$$

The integrals extend over the whole plane ($n = 2$) or straight line ($n = 1$) and may be understood in the usual sense, $T^*(k)$ and $U^*(k)$ are homogeneous of degree 1—the Fourier transforms of $T(x)$ and $U(x)$.

Consider the case $n = 1$ (the two-dimensional problem), letting Ω be the interval $|x|\leqslant 1$ in the (x, y) plane. We will look for a solution of Eqs (1.2) and (1.6) as series

$$
b(x) = \sum_{i=1}^{2N} b^i f(x - x_i),\quad q(x) = \sum_{i=1}^{2N} q^i f(x - x_i),\quad f(x) = \exp\left\{-\frac{x^2}{Dh^2}\right\}
\tag{4.2}
$$

where $x_i = -1 + h(i - 1/2)$ are the interpolation points, $h = 1/N$ is the step length and D is the standard deviation. The solutions are assumed to have this particular form because the action of the operator T of (4.1) on a function $f(x - x_i)$ ($f\in S(\mathbf{R}^1)$) is defined by the fairly simple relationship

$$
(Tf)(x) = A[1 - 2\xi_i \exp(-\xi_i^2)\,\mathrm{Erfi}(\xi_i)],\quad \xi_i^2 = (x - x_i)^2/(h^2 D)
$$

where A is a known constant and $\mathrm{Erfi}(\xi_i)$ is the probability integral of an imaginary argument. The image of $f(x - x_i)$ under U is similar.

Let us consider the approximation (4.2) in detail. Let $u(x)$ be a smooth function whose first and second derivatives are bounded for $x\in(-\infty, \infty)$. We have the following representation.†

$$
u(x) = u_h(x) + R(x),\qquad u_h(x) = \frac{1}{(\pi D)^{1/2}}\sum_{m=-\infty}^{\infty} u(mh)\,f(x - mh)
\tag{4.3}
$$

$$
|R(x)| \leqslant (\|u\| + \|u'\|)R_0(D, h) + \|u''\| h^2 D/4,\quad R_0(D, h) = O(\exp(-\pi^2 D))
\tag{4.4}
$$

where $\|u\|$ is the norm in the space of continuous functions. Thus, when $u(x)$ is approximated by a series $u_h(x)$ as in (4.3), the error depends on two parameters: the standard deviation D and the step length h of the approximation.

† VIL'CHEVSKAYA Ye. N. and KANAUN S. K., Computation of elastic fields in the vicinity of thin inclusions and cracks in a continuous medium. Preprint No. 57, Leningrad Department of the Institute of Mechanical Engineering, Leningrad, 1991.

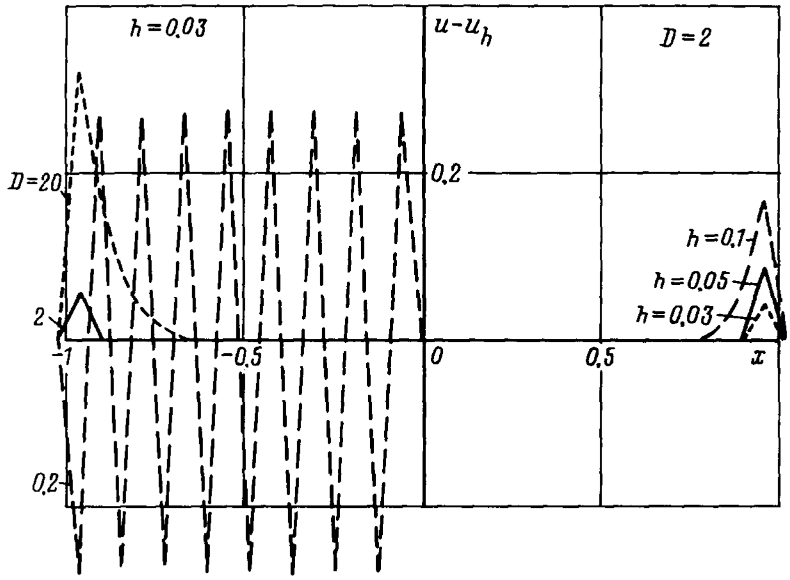

FIG. 1.

Let us see how the approximation (4.3) works for a specific example, defining $u(x)$ as a unit impulse: $u(x)=1$ for $|x|\leqslant 1$, $u(x)=0$ for $|x|>1$. The results for different D and h values are shown in Fig. 1; the left sector of the figure ($x<0$) shows plots of the function $u(x)-u_h(x)$ for a fixed step length $h=1/50$ and different standard deviations D; the right sector shows the analogous plots for fixed $D=2$ and different hs. Obviously, the region of the largest error is concentrated around the points $x=\pm 1$. If h is of the order of 0.03, so that the number of terms remaining on the right of (4.3) is about 60, then the smallest error of the approximation (4.3) is achieved at $D\approx 2$ (in which case $|u(x)-u_h(x)|<0.05$).

We will now proceed to solve Eq. (1.2) using the approximation (4.2). We begin with the two-dimensional problem, again letting Ω be the interval $|x|\leqslant 1$, $y=0$, in the (x,y) plane. For an isotropic medium and a plane deformation the kernel $T(x)$ of T in (1.2) is

$$T^{\alpha\beta}(x) = -\frac{\mu_0\varkappa_0}{\pi x^2}\delta^{\alpha\beta} \tag{4.5}$$

where x^{-2} should be understood as a generalized function whose Fourier transform is $\pi|k|$. If the inclusion is also isotropic, with Lamé coefficients λ and μ and transverse dimensions $h(x)=h_0\alpha(x)$, where $\alpha(x)$ is a non-dimensional function of the shape of the inclusion [$\alpha(x)=0(1)$], then the vector equation (1.2) can be separated into two independent equations:

$$\frac{\Lambda_\alpha b_\alpha(x)}{\alpha(x)} - \frac{\varkappa_0}{\pi}\int_{-\infty}^{\infty}\frac{b_\alpha(x')}{(x-x')^2}dx' = f_\alpha(x), \quad |x|<1 \quad (\alpha=1,2) \tag{4.6}$$

$$\Lambda_1 = \frac{\mu}{h_0\mu_0\varkappa_0}, \quad \Lambda_2 = \frac{\lambda+2\mu}{2h_0\mu_0}, \quad f_\alpha(x) = \frac{\sigma_0^{\alpha 2}(x)}{\mu_0\varkappa_0}$$

These equations take into account that $b(x)=0$ for $|x|>1$. Substituting $b(x)$ from (4.2) into the equations, using (4.1) and requiring the validity of the equations at the interpolation points x_i, we obtain a system of linear algebraic equations for the approximation coefficients b^i:

$$\sum_{i=1}^{2N} A_\alpha^{ki} b_\alpha^{\,i} = f_\alpha^{\,k}, \quad f_\alpha^{\,k} = f_\alpha(x_k), \quad (\alpha=1,2), \quad k=1,\ldots,2N \tag{4.7}$$

$$A_\alpha^{ki} = \frac{\Lambda_\alpha \exp(-\xi_{ik}^2)}{\alpha(x_k)} - \frac{2\pi^{1/2}}{hD^{1/2}}[1-\xi_{ik}\exp(-\xi_{ik}^2)\operatorname{Erfi}(\xi_{ik})], \quad \xi_{ik} = \frac{x_k-x_i}{hD^{1/2}}$$

The matrix A_α^{ki} of this system is completely full, symmetric, and diagonally dominant. The best method for solving system (4.7) is the Seidel method [19].

To improve the accuracy of the computations, it is advisable, using the results of Sec. 3 and [8], to express $b(x)$ as

$$b(x)=\beta(x)(1-x^2)_+^s \tag{4.8}$$

where the second factor on the right reflects the asymptotic behaviour of the solution at the edges $x = \pm 1$ of the inclusion, $f_+ = f$ for $f \geq 0$, $f_+ = 0$ for $f < 0$. The new unknown function is $\beta(x)$. Using the approximation (4.2) for each of the functions $\beta(x)$ and $(1-x^2)_+^s$, we can reduce the problem to the solution of a system similar to (4.7) whose matrix can also be determined by analytical means. Details may be found in the paper cited in the earlier footnote, where the efficiency of the method is also demonstrated by the results of actual computations.

We will now proceed to the three-dimensional problem. For simplicity, we shall consider a crack with the plane surface Ω. If the medium is isotropic, the Fourier transform $T^*(k)$ of $T(x)$ in (1.2) has the following form $[k = k(k_1, k_2)]$:

$$T^{*\alpha\beta}(k)={}^1/_2\mu_0|k|\,[\delta^{\alpha\beta}+\varkappa_0(n^\alpha n^\beta+m^\alpha m^\beta)], \quad m^\alpha=k^\alpha/|k| \tag{4.9}$$

where n is the normal to Ω.

As in the two-dimensional problem, we will look for a solution of Eq. (1.2) with $\lambda(x) = 0$, in the following form (x_1, x_2 are Cartesian coordinates in the plane of the crack):

$$b(x_1, x_2) = \sum_{i=1}^{N} b^i(x_1, x_2) \tag{4.10}$$

$$b^i(x_1, x_2) = b^i \exp\left(-\frac{(x_1-x_{i1})^2}{D_1 h_1^2} - \frac{(x_2-x_{i2})^2}{D_2 h_2^2}\right)$$

where (x_{i1}, x_{i2}) are the coordinates of the interpolation points and h_1 and h_2 are step lengths in the directions of x_1 and x_2. We then choose the standard deviations D_1 and D_2 and step lengths h_1 and h_2 so that $h_1^2 D_1 = h_2^2 D_2 = 4D$. Substitute the series (4.10) into Eq. (1.2) and use the definition (4.1) of T and the expression (4.9) for $T^*(k)$. Requiring the equation to hold true at the interpolation points x_k, we obtain the following system of equations for the coefficients b^i in (4.10):

$$\sum_{i=1}^{N} A_{ki}^{\alpha\beta} b_\beta^i = \sigma_0^{\alpha\beta}(x_k)\, n_\beta, \quad k = 1, \ldots, 2N \tag{4.11}$$

$$A_{ki} = \frac{\mu_0 \pi^{3/2}}{2D^{3/2}} \exp(-\xi_{ki}) \Big\{ 2\,[(1-2\xi_{ki}) J_0(\xi_{ki}) + 2\xi_{ki} J_1(\xi_{ki})]\,(\delta^{\alpha\beta} + \varkappa_0 n^\alpha n^\beta) +$$

$$+ \varkappa_0 \Big[(J_1(\xi_{ki}) - J_0(\xi_{ki}))(e_1^\alpha e_1^\beta + e_2^\alpha e_2^\beta) +$$

$$+ 2\left(J_0(\xi_{ki}) - \left(\frac{1}{2\xi_{ki}} + 1\right) J_1(\xi_{ki})\right) y_{ki}^\alpha y_{ki}^\beta \Big] \Big\}$$

$$y_{ki}^\alpha = (x_{1k}-x_{1i})\, e_1^\alpha + (x_{2k}-x_{2i})\, e_2^\alpha, \quad \xi_{ki} = |y_{ki}|^2/(8D)$$

where e_1 and e_2 are unit vectors in the x_1 and x_2 directions and J_0 and J_1 are Bessel functions.

As in the case of the plane problem, the actual computation of the solution should make allowance for the asymptotic behaviour of $b(x)$ near the edges of the crack, expressing $b(x)$ as $b(x) = \beta(x_1, x_2) \times f_+(x_1, x_2)$, where f_+ is a known function describing the behaviour of the solution as $x \to \Gamma$. Details, including results of actual computations, may be found in the paper cited in the earlier footnote.

To conclude, we point out that the method can also be used to solve Eqs (1.2) and (1.6) for a non-planar surface Ω. The elements of the matrix of the system of linear algebraic equations to which the original problem is reduced may then be found in analytical form, though the latter is more cumbersome than (4.11).

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